

# TOWARDS A CLASSIFICATION OF HOMOGENEOUS TUBE DOMAINS IN $\mathbb{C}^4$

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**ABSTRACT.** We classify the tube domains in  $\mathbb{C}^4$  with affinely homogeneous base whose boundary contains a non-degenerate affinely homogeneous hypersurface. It follows that these domains are holomorphically homogeneous and amongst them there are four new examples of unbounded homogeneous domains (that do not have bounded realisations). These domains lie to either side of a pair of Levi-indefinite hypersurface. Using the geometry of these two hypersurfaces, we find the automorphism groups of the domains.

## 0. INTRODUCTION

The study of holomorphically homogeneous domains in complex space goes back to É. Cartan [C] who determined all bounded symmetric domains in  $\mathbb{C}^n$  as well as all bounded homogeneous domains in  $\mathbb{C}^2$  and  $\mathbb{C}^3$ . Due to the fundamental theorem of Vinberg, Gindikin, and Pyatetskii-Shapiro, every bounded homogeneous domains can be realised as a Siegel domain of the second kind (see [P-S]). Although this result does not immediately imply a complete classification of bounded homogeneous domains, it reduces the classification problem to that for domains of a very special form. A generalisation of the above theorem to the case of unbounded domains for the class of rational homogeneous domains was obtained by Penney in his remarkable paper [P2], where the role of models is played by so-called Siegel domains of type  $N-P$ . Nevertheless, the classification problem for the unbounded case is far from fully understood.

In [L2], Loboda shows that any holomorphically homogeneous non-spherical tube hypersurface in  $\mathbb{C}^2$  has an affinely homogeneous base. The generalisation of this result to higher dimensions is an open problem. Loboda's result motivates our study. We shall consider tubes over

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affinely homogeneous bases and suppose that the boundary of the base contains an affinely homogeneous hypersurface. In these circumstances, we can classify the resulting tube domains in low dimensions and determine their automorphism groups. The first interesting examples occur in  $\mathbb{C}^4$ .

A simple natural class of unbounded homogeneous domains comes from generalising the unbounded form of the unit ball

$$B^n := \{z \in \mathbb{C}^n : \operatorname{Re} z_n > |z'|^2\}$$

and the complementary domain  $\{z \in \mathbb{C}^n : \operatorname{Re} z_n < |z'|^2\}$ , where  $z = (z_1, \dots, z_n)$  are coordinates in  $\mathbb{C}^n$  and  $z' := (z_1, \dots, z_{n-1})$ . Indeed, the domains lying on either side of the quadric given by the equation

$$(0.1) \quad \operatorname{Re} z_n = \langle z', z' \rangle,$$

where  $\langle z', z' \rangle$  is a Hermitian form in  $\mathbb{C}^{n-1}$ , are easily seen to be homogeneous. Among them,  $B^n$  is the only domain, up to a linear change of coordinates, that admits a bounded realisation. Domains of this kind with  $\langle z', z' \rangle$  non-degenerate were generalised by Penney, who introduced and studied a special class of Siegel  $N$ - $P$  domains called (homogeneous) *nil-balls* (see [P1, P3]). A nil-ball is a certain  $N$ - $P$  domain on which a solvable algebraic group acts transitively and polynomially by biholomorphic transformations, with smooth connected algebraic Levi non-degenerate boundary on which a codimension 1 nil-radical of the group acts simply transitively.

The examples arising from Hermitian forms, as above, can also be written as tube domains, i.e. domains of the form  $D_\Omega := \Omega + i\mathbb{R}^n$ , where the base  $\Omega$  is a domain in  $\mathbb{R}^n \subset \mathbb{C}^n$  (see, e.g., [I] for an explicit change of coordinates). Tube domains are clearly unbounded and often do not have bounded realisations. If  $\Omega$  is affinely homogeneous, then  $D_\Omega$  is holomorphically homogeneous since every affine mapping of  $\mathbb{R}^n$  can be lifted to an affine mapping of  $\mathbb{C}^n$  and since  $D_\Omega$  is invariant under translations in imaginary directions. In [P1] a large class of nil-balls of this kind is introduced.

The new examples of tube domains in  $\mathbb{C}^4$  that we give in this paper have affinely homogeneous bases, possess no bounded realisations, and are not equivalent to nil-balls. They arise by the following construction/classification. Start with an affinely homogeneous hypersurface  $\Gamma \subset \mathbb{R}^n$ . We might find that the orbits of the affine symmetry group of  $\Gamma$  are, in addition to  $\Gamma$  itself, some domain or domains  $\Omega$  to either side of it. Of course, in this case, the action of the symmetry group on  $\Gamma$  must have isotropy. A classification of suitable  $\Gamma$  and  $\Omega$  leads to a classification of corresponding homogeneous tubes  $D_\Omega$ .

Affinely homogeneous hypersurfaces  $\Gamma \subset \mathbb{R}^n$  have been classified for  $n = 2$  (see, e.g., [NS1]) and  $n = 3$  in [DKR, EE1]. We are interested in hypersurfaces with non-trivial affine isotropy subgroup, and when  $n = 4$  such  $\Gamma$  have also been classified (see [EE2]) under the additional hypothesis of non-degeneracy (equivalently, under the hypothesis that  $\Gamma + i\mathbb{R}^n$  is Levi non-degenerate). In this paper we concentrate on the case of non-degenerate  $\Gamma$  for  $n = 4$ . This leads to a relatively small number of affinely homogeneous domains: most of the automorphism groups from [EE2] do not have open orbits. This, in turn, leads to the following classification in  $\mathbb{C}^4$ :-

**Theorem 0.1.** *Suppose  $D_\Omega = \Omega + i\mathbb{R}^4$  is a tube with affinely homogeneous base, having as part of its boundary an affinely homogeneous non-degenerate hypersurface. Then, up to biholomorphism,  $D_\Omega$  must be exactly one of the following:-*

$$\begin{aligned}
B_+^> &:= \{z \in \mathbb{C}^4 : x_4 > x_1^2 + x_2^2 + x_3^2\}, \\
B_+^< &:= \{z \in \mathbb{C}^4 : x_4 < x_1^2 + x_2^2 + x_3^2\}, \\
B_-^> &:= \{z \in \mathbb{C}^4 : x_4 > x_1^2 + x_2^2 - x_3^2\}, \\
B_-^< &:= \{z \in \mathbb{C}^4 : x_4 < x_1^2 + x_2^2 - x_3^2\}, \\
H^> &:= \{z \in \mathbb{C}^4 : x_4 > x_1x_2 + x_3^2 \text{ and } x_1 > 0\}, \\
H^< &:= \{z \in \mathbb{C}^4 : x_4 < x_1x_2 + x_3^2 \text{ and } x_1 > 0\}, \\
N_+^> &:= \{z \in \mathbb{C}^4 : x_4 > x_1x_2 + x_3^2 + x_1^2x_3 + x_1^4\}, \\
N_+^< &:= \{z \in \mathbb{C}^4 : x_4 < x_1x_2 + x_3^2 + x_1^2x_3 + x_1^4\}, \\
N_-^> &:= \{z \in \mathbb{C}^4 : x_4 > x_1x_2 + x_3^2 + x_1^2x_3 - x_1^4\}, \\
N_-^< &:= \{z \in \mathbb{C}^4 : x_4 < x_1x_2 + x_3^2 + x_1^2x_3 - x_1^4\}, \\
C^> &:= \{z \in \mathbb{C}^4 : x_4 > x_1x_2 + x_1x_3^2 \text{ and } x_1 > 0\}, \\
C^< &:= \{z \in \mathbb{C}^4 : x_4 < x_1x_2 + x_1x_3^2 \text{ and } x_1 > 0\}, \\
D^> &:= \{z \in \mathbb{C}^4 : x_4^2 > x_1x_2 + x_1^2x_3 \text{ and } x_1 > 0\}, \\
D^< &:= \{z \in \mathbb{C}^4 : x_4^2 < x_1x_2 + x_1^2x_3 \text{ and } x_1 > 0\},
\end{aligned}$$

where  $x_j = \operatorname{Re} z_j$  for  $j = 1, \dots, 4$ .

The domain  $B_+^>$  is the ordinary unit ball written in tube form. The domains  $B_+^<$ ,  $B_-^>$ ,  $B_-^<$  are the tube realisations of domains lying to either side of a quadric (0.1) as mentioned earlier. They are the (pseudo-) balls and, in the indefinite case, there are also affinely homogeneous domains  $H^>$  and  $H^<$ , which can be thought of as ‘half-pseudo-balls’. The domains  $N_+^>$ ,  $\dots$ ,  $N_-^<$  are nil-balls in the sense of Penney [P1, P3]. This leaves  $C^>$ ,  $C^<$ ,  $D^>$ ,  $D^<$ , which are new. Remarkably, for each of these domains, the Levi non-degenerate part of the boundary admits an explicit rational transformation to Chern-Moser normal form. In this normal form the CR isotropy becomes linear. This enables us to

determine the full holomorphic automorphism group of the domains in question.

The only pseudo-convex domain in Theorem 0.1 is the unit ball  $B_+^>$ . Its complement,  $B_+^<$  is pseudo-concave. The remaining domains naturally fall into two classes

$$B_-^>, H^>, N_+^>, N_-^>, C^>, D^> \quad \text{versus} \quad B_-^<, H^<, N_+^<, N_-^<, C^<, D^<$$

according to whether the Levi form of the non-degenerate part of the boundary has type  $++-$  or  $+-$  (in some cases there is also a Levi flat part to the boundary).

The paper is organised as follows. In Section 1 we examine the list of [EE2] and use it to prove Theorem 0.1. The new domains  $D^>$  and  $D^<$  are the main subject of the remainder of the article. We verify that they do not have bounded realisations. In Section 2 we determine the automorphism groups of these domains by means of exploiting the properties of  $\Gamma$ . These automorphism groups turn out to be Lie groups of dimension 10, and their structure shows that the domains that  $D^>$  and  $D^<$  are not equivalent to any nil-balls. The new domains  $C^>$  and  $C^<$  are similar but slightly easier. In Section 3 we sketch their analysis along the lines already done in Section 2 for  $D^>$  and  $D^<$ .

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## 1. CLASSIFICATION AND NEW EXAMPLES

The non-degenerate affinely homogeneous hypersurfaces with isotropy in  $\mathbb{R}^4$  are classified in [EE2], where they are grouped according to their complexification into 20 different types, some of which have more than one real form. Sometimes, there is a parameter, as in hypersurface #4:–

$$\{x \in \mathbb{R}^4 : x_4 = x_1x_2 + x_3^2 + x_1^2x_3 + \alpha x_1^4\}, \quad \text{for } \alpha \in \mathbb{R},$$

where  $x := (x_1, x_2, x_3, x_4)$ . There is a list of explicit defining equations in [EE2] and it is an elementary matter to verify that each hypersurface  $\Gamma$  from the list is, indeed, affinely homogeneous with isotropy. This is best done by infinitesimal means: attached to any hypersurface is its symmetry algebra, namely the affine vector fields on  $\mathbb{R}^4$  that are tangent to  $\Gamma$  along  $\Gamma$ , and the question is whether this algebra is acting transitively near  $\Gamma$ 's chosen basepoint and has dimension at least 4. This is mere computation and in many cases, the symmetry algebra is exactly 4-dimensional. In these cases, we may manufacture a  $4 \times 4$  matrix from any chosen basis of affine vector fields and compute its determinant: it

will be a polynomial on  $\mathbb{R}^4$ . Where this polynomial does not vanish, the symmetry algebra contains linearly independent vector fields and so there are open orbits. Otherwise there cannot be open orbits. All of these computations are easily carried out with computer algebra and a Maple program, with the explicit defining equations already included, is available<sup>1</sup>. It turns out that this determinant is, in many cases, identically zero. Hypersurfaces #7–#20 are thereby eliminated and we are left with (after some minor affine changes of variables from [EE2]):–

	Equation	Basepoint
#1	$x_4 = x_1^2 + x_2^2 \pm x_3^2$	(0, 0, 0, 0)
#2	$x_1^2 + x_2^2 + x_3^3 \pm x_4^2 = 1$ or $x_1^2 - x_2^2 - x_3^3 \pm x_4^2 = 1$	(1, 0, 0, 0)
#3	$x_4 = x_1x_2 + x_3^2 + x_1^3$	(0, 0, 0, 0)
#4	$x_4 = x_1x_2 + x_3^2 + x_1^2x_3 + \alpha x_1^4$	(0, 0, 0, 0)
#5	$x_4 = x_1x_2 + x_1x_3^2$	(1, 0, 0, 0)
#6	$x_4^2 = x_1x_2 + x_1^2x_3$	(1, 0, 1, 1)

*Proof of Theorem 0.1:* We have just identified the only possible affinely homogeneous hypersurfaces satisfying the hypotheses of Theorem 0.1. That is not to say that all these possibilities actually occur and, in fact, #2 does not. The affine symmetry algebra of the sphere, for example, is spanned by the six vector fields  $x_j\partial/\partial x_k - x_k\partial/\partial x_j$  for  $j < k$  and these fields are nowhere linearly independent. In fact, the associated group is  $\text{SO}(4)$  with orbits concentric spheres and the origin. A similar conclusion holds for other signatures.

All the remaining cases do, in fact, give rise to affinely homogeneous domains as follows. Case #1, with either sign, has a 7-dimensional symmetry algebra  $\mathcal{S}$  and outside of  $\Gamma$  itself these fields have maximal rank: we obtain  $B_+^>, B_+^<, B_-^>, B_-^<$ . There is also the possibility of having a Lie subalgebra  $\mathcal{R} \subset \mathcal{S}$  with  $\dim \mathcal{R} = 4, 5, 6$  and such that the fields in  $\mathcal{R}$  have maximal rank somewhere but not everywhere outside  $\Gamma$ . This possibility is easily investigated with computer algebra. First one finds the possible subalgebras by parameterising the Grassmannians  $\text{Gr}_k(\mathcal{S})$  for  $k = 4, 5, 6$  with affine coordinate patches in the usual way, in each case solving the equations that these subspaces be closed under Lie bracket. Then one computes the determinants of the  $4 \times 4$  minors for each of these subalgebras. Up to affine change of coordinates we obtain just two possible affinely homogeneous ‘half-pseudo-balls’  $H^>$  and  $H^<$ , these domains having 5-dimensional affine symmetry.

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<sup>1</sup>[ftp://ftp.maths.adelaide.edu.au/pure/meastwood/maple7/orbits](http://ftp.maths.adelaide.edu.au/pure/meastwood/maple7/orbits)

Case #3 has a 5-dimensional symmetry algebra and the domains to either side of  $\Gamma$  are easily verified to be affinely homogeneous. It is a simple computation to check that all of its 4-dimensional subalgebras consist of fields that are nowhere linearly independent. So, case #3 gives precisely the following two holomorphically homogeneous domains

$$\{x \in \mathbb{R}^4 : x_4 > x_1x_2 + x_3^2 + x_1^3\} + i\mathbb{R}^4, \{x \in \mathbb{R}^4 : x_4 < x_1x_2 + x_3^2 + x_1^3\} + i\mathbb{R}^4.$$

However, the polynomial transformation

$$z_2 \mapsto z_2 - 3z_1^2/2 \quad z_4 \mapsto z_4 - z_1^3/2$$

takes these two domains biholomorphically onto domains that we have already listed, namely  $B^>$  and  $B^<$ , respectively.

Case #4 is the subject of [EI] and will appear elsewhere. In brief, there are three subcases according to whether  $\alpha > 1/12$ ,  $\alpha = 1/12$ , or  $\alpha < 1/12$ . The case  $\alpha = 1/12$  again yields  $B^>$  and  $B^<$  by an explicit polynomial change of coordinates. The equations of the other hypersurfaces may be placed in Chern-Moser normal form [CM]:–

$$2\operatorname{Im} w_4 = w_1\overline{w_2} + w_2\overline{w_1} + |w_3|^2 \pm |w_1|^4,$$

where the sign before  $|w_1|^4$  is the sign of  $\alpha - 1/12$ . This sign is a CR invariant (as observed, for example, in [BSW]) and we obtain the domains  $N^>_+$ ,  $N^<_+$ ,  $N^>_-$ ,  $N^<_-$ . These were also found by Penney [P3] and are examples of his *nil-balls*.

In cases #4, #5, and #6, the symmetry algebra is 4-dimensional and so there is no possibility of a Lie subalgebra giving rise to another domain. The Maple computer program ‘orbits’ mentioned early in this section finds the only open orbits and these account for the remaining domains  $C^>$ ,  $C^<$ ,  $D^>$ ,  $D^<$  in the statement of Theorem 0.1. That they are holomorphically distinct and not nil-balls is proved in Section 2, concerned with  $D^>$  and  $D^<$ , and Section 3, concerned with  $C^>$  and  $C^<$ . With these matters postponed, the theorem is proved.  $\square$

We conclude this section with some preliminary discussion of the cases  $D^>$  and  $D^<$  before a detailed analysis of their holomorphic automorphisms in Section 2. So, consider the following affinely homogeneous hypersurface in  $\mathbb{R}^4$ :–

$$\Gamma := \{x \in \mathbb{R}^4 : x_4^2 = x_1x_2 + x_1^2x_3, x_1 > 0\},$$

and let

$$\begin{aligned} \Omega^> &:= \{x \in \mathbb{R}^4 : x_4^2 > x_1x_2 + x_1^2x_3, x_1 > 0\}, \\ \Omega^< &:= \{x \in \mathbb{R}^4 : x_4^2 < x_1x_2 + x_1^2x_3, x_1 > 0\}. \end{aligned}$$

Clearly,  $\Gamma \subset \partial\Omega^> \cap \partial\Omega^<$ . In fact, these domains are not as similar as they might first appear. Part of the boundary of  $\Omega^>$  is Levi flat,

namely  $\{x \in \mathbb{R}^4 : x_1 = 0 \text{ and } x_4 \neq 0\}$ . In the case of  $\Omega^<$ , however, the inequality  $x_1 > 0$  serves only to specify one of the two connected components of  $\{x \in \mathbb{R}^4 : x_4^2 < x_1x_2 + x_1^2x_3\}$ .

One can verify that  $\Omega^>$  and  $\Omega^<$  are invariant under the group  $G$  that consists of the following affine transformations of  $\mathbb{R}^4$

$$\begin{aligned} x_1 &\mapsto qx_1, \\ x_2 &\mapsto qr^2(s + t^2)x_1 + qr^2x_2 + 2qr^2tx_4, \\ x_3 &\mapsto r^2x_3 - r^2s, \\ x_4 &\mapsto qrtx_1 + qrx_4, \end{aligned}$$

where  $q > 0$ ,  $r \in \mathbb{R}^*$ ,  $s, t \in \mathbb{R}$ . We will now show that  $G$  acts transitively on each of  $\Omega^>$  and  $\Omega^<$ . Take the point  $(1, 0, 0, 1) \in \Omega^>$  and apply a mapping from  $G$  to it. The result is the point

$$(q, qr^2(t^2 + 2t + s), -r^2s, qr(t + 1)).$$

Let  $(x_1^0, x_2^0, x_3^0, x_4^0)$  be any other point in  $\Omega^>$ . Then setting

$$\begin{aligned} q &= x_1^0, & r &= \frac{\sqrt{(x_4^0)^2 - x_1^0x_2^0 - (x_1^0)^2x_3^0}}{x_1^0}, \\ t &= \frac{x_4^0}{\sqrt{(x_4^0)^2 - x_1^0x_2^0 - (x_1^0)^2x_3^0}} - 1, & s &= -\frac{(x_1^0)^2x_3^0}{(x_4^0)^2 - x_1^0x_2^0 - (x_1^0)^2x_3^0}, \end{aligned}$$

we obtain an element of  $G$  that maps  $(1, 0, 0, 1)$  into  $(x_1^0, x_2^0, x_3^0, x_4^0)$ . This proves that  $\Omega^>$  is affinely homogeneous. A similar argument shows that  $\Omega^<$  is affinely homogeneous as well. It can be shown that  $G$  is in fact the full group of affine automorphisms of each of  $\Omega^>$  and  $\Omega^<$ .

It then follows that the corresponding tube domains  $D^> := D_{\Omega^>}$  and  $D^< := D_{\Omega^<}$  are holomorphically homogeneous since the group  $\tilde{G}$  generated by  $G$  (viewed as a group of affine transformations of  $\mathbb{C}^4$ ) and translations in the imaginary directions in  $\mathbb{C}^4$ , acts transitively on each of them.

We note that neither of these domains has a bounded realisation. In fact, neither of them is Kobayashi-hyperbolic (we remark here that it is shown in [N] that any connected homogeneous Kobayashi-hyperbolic manifold is biholomorphically equivalent to a bounded domain in complex space). Indeed, the domain  $D^>$  contains the affine complex line  $\{z \in \mathbb{C}^4 : z_1 = 1, z_2 + z_3 = 0, z_4 = 1\}$  while  $D^<$  contains the affine complex line  $\{z \in \mathbb{C}^4 : z_1 = 1, z_2 + z_3 = 1, z_4 = 0\}$ .

## 2. THE AUTOMORPHISM GROUPS OF $D^>$ AND $D^<$

Denote by  $G^>$  and  $G^<$  the groups of holomorphic automorphisms of  $D^>$  and  $D^<$  respectively, equipped with the compact-open topology. In this section we will determine these groups.

Let  $\tilde{\Gamma} := \Gamma + i\mathbb{R}^4$ . Clearly,  $\tilde{\Gamma}$  is a connected Levi non-degenerate hypersurface contained in  $\partial D^> \cap \partial D^<$ . Since the Levi form of  $\tilde{\Gamma}$  at every point has a positive and a negative eigenvalue, every element of  $G^>$  and  $G^<$  extends past  $\tilde{\Gamma}$  to a biholomorphic mapping preserving  $\tilde{\Gamma}$ . Let  $H$  be the subgroup of  $\tilde{G}$  given by the condition  $r = 1$ . It is straightforward to verify that  $H$  acts simply transitively on  $\tilde{\Gamma}$ . Therefore, every element  $g$  of either  $G^>$  or  $G^<$ , for a fixed  $p \in \tilde{\Gamma}$ , can be uniquely represented as  $g = h \circ t$ , where  $h \in H$  and  $t$  is an element of either  $I_p^>$  or  $I_p^<$ , and  $I_p^>$  and  $I_p^<$  denote the isotropy subgroups of  $p$  in  $G^>$  and  $G^<$  respectively.

Thus, in order to determine  $G^>$  and  $G^<$  we must find  $I_p^>$  and  $I_p^<$  for some  $p \in \tilde{\Gamma}$ . Let  $p_0 := (1, 0, 1, 1)$  and  $I_{p_0}$  be the group of all *local* holomorphic automorphisms of  $\tilde{\Gamma}$  defined near  $p_0$  and preserving  $p_0$ . Clearly,  $I_{p_0}^> \subset I_{p_0}$  and  $I_{p_0}^< \subset I_{p_0}$ . Equipped with the topology of uniform convergence of the derivatives of all orders of the component functions on compact subsets of  $p_0$ , the group  $I_{p_0}$  is known to carry the structure of a real algebraic group (see, e.g., [B], [L1], [VK]).

It is straightforward to show that  $I_{p_0} \cap \tilde{G}$  consists of the mappings

$$\begin{aligned} z_1 &\mapsto z_1, \\ z_2 &\mapsto 2(r^2 - r)z_1 + r^2z_2 + 2(r - r^2)z_4, \\ z_3 &\mapsto r^2z_3 + 1 - r^2, \\ z_4 &\mapsto (1 - r)z_1 + rz_4, \end{aligned}$$

where  $r \in \mathbb{R}^*$ . As the following proposition shows,  $I_{p_0}$  is in fact 3-dimensional and admits an explicit description.

**Proposition 2.1.** *The group  $I_{p_0}$  consists of all mappings of the form*

$$(2.1) \quad \begin{aligned} z_1 &\mapsto z_1, \\ z_2 &\mapsto -2v^2z_1^3 + i(-2v + 4vr + u)z_1^2 - 4ivr z_1 z_4 \\ &\quad + 2(v^2 + r^2 - r)z_1 + r^2z_2 + 2(r - r^2)z_4 + 2iv - iu, \\ z_3 &\mapsto 2v^2z_1^2 - i(2u + 4vr)z_1 + r^2z_3 + 4ivr z_4 - r^2 \\ &\quad + 1 - 2v^2 + 2iu, \\ z_4 &\mapsto -ivz_1^2 + (1 - r)z_1 + rz_4 + iv, \end{aligned}$$

where  $r \in \mathbb{R}^*$ ,  $u, v \in \mathbb{R}$ .



*Proof:* First, we will write the equation of  $\tilde{\Gamma}$  near  $p_0$  in Chern-Moser normal form [CM]. Consider the change of coordinates

$$\begin{aligned} z_1 &= \frac{11w_1 + 4}{w_1 + 4}, \\ z_2 &= -\frac{96i(4iw_2 - 5iw_3 + 11w_4)}{5w_1 + 20} \\ &\quad + \frac{1600i(4iw_2 - 5iw_3 - 6iw_3^2 + 6w_4)}{(5w_1 + 20)^2} - \frac{1280w_3^2}{(w_1 + 4)^3} + 24iw_4, \\ z_3 &= \frac{32i(2iw_2 + 3w_4)}{5w_1 + 20} - \frac{32w_3^2}{(w_1 + 4)^2} - 4iw_4 + 1, \\ z_4 &= -\frac{8(6w_3 + 5)}{w_1 + 4} + \frac{160w_3}{(w_1 + 4)^2} + 11. \end{aligned}$$

In the  $w$ -coordinates the point  $p_0$  becomes the origin, and the equation of  $\tilde{\Gamma}$  near the origin takes the form

$$(2.2) \quad \text{Im } w_4 = F(w', \overline{w'}) := \frac{N(w', \overline{w'})}{D(w', \overline{w'})},$$

where  $w' := (w_1, w_2, w_3)$  and

$$\begin{aligned} D(w', \overline{w'}) &= (11|w_1|^2 + 24w_1 + 24\overline{w_1} + 16)(5|w_1|^2 + 16), \\ N(w', \overline{w'}) &= 8\text{Re} \left( 48w_1|w_3|^2 + 25w_1^2\overline{w_3}^2 + 16|w_3|^2 + 36|w_1|^2|w_3|^2 \right. \\ &\quad \left. + 2(11|w_1|^2 + 24w_1 + 24\overline{w_1} + 16)w_1\overline{w_2} \right). \end{aligned}$$

Denote by  $F_{k\bar{l}}$  the polynomial of degree  $k$  in  $w'$  and degree  $l$  in  $\overline{w'}$  in the power series expansion of the function  $F(w', \overline{w'})$  about the origin. A straightforward calculation gives

$$\begin{aligned} F_{1\bar{1}} &= \frac{|w_3|^2}{2} + \text{Re } w_1\overline{w_2}, \\ F_{2\bar{2}} &= 2\text{Re} \left( -\frac{5}{32}w_1^2\overline{w_1}w_2 + \frac{25}{64}w_1^2\overline{w_3}^2 + \frac{5}{16}|w_1|^2|w_3|^2 \right), \\ F_{3\bar{2}} &= -\frac{75}{128}w_1^3\overline{w_3}^2 - \frac{75}{64}w_1^2w_3\overline{w_1}w_3 - \frac{75}{128}w_1w_3^2\overline{w_1}^2, \\ F_{3\bar{3}} &= 2\text{Re} \left( \frac{25}{512}w_1^3\overline{w_1}^2\overline{w_2} + \frac{175}{128}w_1^3\overline{w_1}w_3^2 + \frac{1425}{1024}|w_1|^4|w_3|^2 \right). \end{aligned} \tag{2.3}$$

It follows from the first formula in (2.3) that the operator  $tr$  defined in [CM] in this case is

$$tr = 2 \left( \frac{\partial^2}{\partial w_1 \partial \bar{w}_2} + \frac{\partial^2}{\partial w_2 \partial \bar{w}_1} + \frac{\partial^2}{\partial w_3 \partial \bar{w}_3} \right).$$

Equation (2.2) and the last three formulae in (2.3) now yield

$$F(w', 0) = \frac{\partial F}{\partial w'}(w', 0) = tr F_{2\bar{2}} = tr^2 F_{3\bar{2}} = tr^2 F_{3\bar{3}} = 0,$$

which shows that equation (2.2) is indeed in Chern-Moser normal form.

We will now find  $I_{p_0}$  in the  $w$ -coordinates. Let  $I$  denote the group of mappings of the form

$$\begin{aligned} w_1 &\mapsto w_1, \\ w_2 &\mapsto \left( i\mu - \frac{\nu^2}{2} \right) w_1 + r^2 w_2 + i\nu r w_3, \\ w_3 &\mapsto i\nu w_1 + r w_3, \\ w_4 &\mapsto r^2 w_4, \end{aligned}$$

where  $r \in \mathbb{R}^*$ ,  $\mu, \nu \in \mathbb{R}$ . One can directly verify that  $I$  is a subgroup of  $I_{p_0}$ . We will show that in fact  $I_{p_0} = I$ .

Suppose that  $I_{p_0} \neq I$  and let  $f \in I_{p_0} \setminus I$ . It then easily follows from the condition that the term  $F_{2\bar{2}}$  does not change when  $f$  is applied to equation (2.2) that, without loss of generality, the Jacobian matrix of  $f$  at the origin can be assumed to have the form

$$(2.4) \quad \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where  $|\beta| = 1$ ,  $\beta \neq 1$ . Since the local isotropy group of any Levi non-degenerate hypersurface in  $\mathbb{C}^4$  is linearisable (see [E1], [E2]), there exist normal coordinates in which transformations from  $I$  remain linear and  $f$  coincides with the linear transformation given by matrix (2.4). Since the term  $F_{3\bar{2}}$  in these new coordinates has to be preserved by  $f$ , it follows that  $F_{3\bar{2}} = 0$ . On the other hand, one can show that the term  $F_{3\bar{2}}$  in formula (2.3) can not be eliminated by any transformation to the normal form that leaves all mappings from  $I$  linear. This follows from the fact that the family of Chern-Moser chains invariant under the action of  $I$  is characterised by the condition that the tangent vector of a chain at the origin is a multiple of a vector of the form  $(0, a, 0, 1)$ . Thus, for a normalisation  $w \mapsto w^*$  that transforms such a chain into the line  $\{w^{*'} = 0, \operatorname{Im} w_4^* = 0\}$ , we have  $\partial w^{*'} / \partial w_4(0) = (0, a, 0, 1)$ . With this

single parameter freedom one can not eliminate the entire term  $F_{3\bar{2}}$  in formula (2.3). This contradiction shows that  $I_{p_0} = I$ .

Recomputing the group  $I_{p_0}$  in the  $z$ -coordinates, we obtain formulae (2.1) with  $u = \frac{16}{25}\mu$  and  $v = \frac{2}{5}\nu$ . The proposition is proved.  $\square$

Since all elements of  $I_{p_0}$  are polynomial mappings, we have  $I_{p_0} \subset I_{p_0}^>$  and  $I_{p_0} \subset I_{p_0}^<$ , which implies that  $I_{p_0}^> = I_{p_0}^< = I_{p_0}$ . Hence  $G^> = G^< = H \circ I_{p_0}$ . A straightforward calculation now leads to the following

**Theorem 2.2.** *The automorphism groups of  $D^>$  and  $D^<$  coincide and consist of all mappings of the form*

$$\begin{aligned}
 (2.5) \quad \begin{aligned}
 z_1 &\mapsto qz_1 + i\alpha_1, \\
 z_2 &\mapsto -2qv^2z_1^3 + iq(-2v + 4vr - 2vt + u)z_1^2 - 4iqvrz_1z_4 \\
 &\quad + q(2v^2 + 2r^2 - 2r - 2tr + 2t + s + t^2)z_1 + qr^2z_2 \\
 &\quad + 2q(r - r^2 + tr)z_4 + i\alpha_2, \\
 z_3 &\mapsto 2v^2z_1^2 - 2i(2vr + u)z_1 + r^2z_3 + 4ivrz_4 \\
 &\quad - r^2 - 2v^2 - s + 1 + i\alpha_3, \\
 z_4 &\mapsto -iqvz_1^2 + q(t - r + 1)z_1 + qrz_4 + i\alpha_4,
 \end{aligned}
 \end{aligned}$$

where  $s, t, u, v, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ ,  $q > 0$ ,  $r \in \mathbb{R}^*$ .

We will now show that neither of  $D^>$ ,  $D^<$  is equivalent to a nil-ball. Otherwise one of these domains would be holomorphically equivalent to a domain  $D$  that has a Levi non-degenerate boundary and admits an action of a connected nilpotent Lie group  $N$  by holomorphic transformations defined in a neighbourhood of  $\bar{D}$ , such that the induced action of  $N$  on  $\partial D$  is transitive (see [P1], [P3]). Let  $F$  denote the equivalence mapping. Since the Levi form of  $\tilde{\Gamma}$  at every point has a positive and a negative eigenvalue,  $F$  extends to a neighbourhood of  $\tilde{\Gamma}$  to a biholomorphic mapping onto a neighbourhood of  $\bar{D}$ , and we have  $F(\tilde{\Gamma}) = \partial D$  (the values of  $F$  near the part of the boundary intersecting the hyperplane  $\{\operatorname{Re} z_1 = 0\}$  necessarily approach infinity). Therefore,  $N$  acts on either  $D^>$  or  $D^<$  by biholomorphic transformations defined in a neighbourhood of  $\tilde{\Gamma}$  in such a way that the induced action of  $N$  on  $\tilde{\Gamma}$  is transitive. This implies that the Lie algebra  $\mathfrak{g}$  of  $G^> = G^<$  has a nilpotent subalgebra that acts transitively on  $\tilde{\Gamma}$ . We will now show that such a subalgebra in fact does not exist.

The algebra  $\mathfrak{g}$  is spanned by the following holomorphic vector fields

$$\begin{aligned}
Z_1 &:= z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + z_4 \frac{\partial}{\partial z_4}, \quad Z_2 := z_1 \frac{\partial}{\partial z_2} - \frac{\partial}{\partial z_3}, \\
Z_3 &:= 2z_4 \frac{\partial}{\partial z_2} + z_1 \frac{\partial}{\partial z_4}, \quad Z_4 := i \frac{\partial}{\partial z_1} + i \frac{\partial}{\partial z_4}, \\
Z_5 &:= i \frac{\partial}{\partial z_2}, \quad Z_6 := i \frac{\partial}{\partial z_3}, \quad Z_7 := 2i \frac{\partial}{\partial z_2} + i \frac{\partial}{\partial z_4}, \\
Z_8 &:= 2(z_1 + z_2 - z_4) \frac{\partial}{\partial z_2} + 2(z_3 - 1) \frac{\partial}{\partial z_3} + (z_4 - z_1) \frac{\partial}{\partial z_4}, \\
Z_9 &:= iz_1^2 \frac{\partial}{\partial z_2} - 2iz_1 \frac{\partial}{\partial z_3}, \\
Z_{10} &:= 2i(z_1^2 - 2z_1 z_4) \frac{\partial}{\partial z_2} - 4i(z_1 - z_4) \frac{\partial}{\partial z_3} - iz_1^2 \frac{\partial}{\partial z_4}.
\end{aligned}$$

The commutation relations for the vector fields above are as follows

$[Z_i, Z_j]$	$Z_1$	$Z_2$	$Z_3$	$Z_4$	$Z_5$	$Z_6$	$Z_7$	$Z_8$	$Z_9$	$Z_{10}$
$Z_1$		0	0	$-Z_4$	$-Z_5$	0	$-Z_7$	0	$Z_9$	$Z_{10}$
$Z_2$			0	$-Z_5$	0	0	0	$2Z_2$	0	0
$Z_3$				$-Z_7$	0	0	$-2Z_5$	$Z_3$	0	$-2Z_9$
$Z_4$					0	0	0	0	$-2Z_2$	$2Z_3$
$Z_5$						0	0	$2Z_5$	0	0
$Z_6$							0	$2Z_6$	0	0
$Z_7$								$Z_7$	0	$4Z_2$
$Z_8$									$-2Z_9$	$-Z_{10}$
$Z_9$										0
$Z_{10}$										

If a subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  acts transitively on  $\tilde{\Gamma}$ , it must contain vector fields  $Z'_1 := Z - Z_1$ ,  $Z'_4 := Z_4 + W$ , with some  $Z, W \in \mathfrak{i}$ , where  $\mathfrak{i}$  is the Lie algebra of  $I_{p_0}$  (observe that  $\mathfrak{i}$  is spanned by  $Z_8, Z_9 - Z_5 + 2Z_6, Z_{10} + Z_7$ ). It follows from the commutation relations above that  $[Z'_1, Z'_4]$  is a vector field of the form  $Z_4 + L$ , where  $L$  is a linear combination of  $Z_2, Z_3, Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}$ . Thus, by induction, all the higher-order commutators  $[Z'_1, [Z'_1, \dots, [Z'_1, [Z'_1, Z'_4]] \dots]]$  have this form and hence are non-zero. Therefore,  $\mathfrak{h}$  is not nilpotent.

Thus, we have proved the following:–

**Theorem 2.3.** *Each of the domains  $D^>, D^< \subset \mathbb{C}^4$  is holomorphically homogeneous, does not have a bounded realisation, and is not equivalent to a nil-ball.*

### 3. THE AUTOMORPHISM GROUPS OF $C^>$ AND $C^<$

The discussion for  $C^>$  and  $C^<$  largely parallels that for  $D^>$  and  $D^<$  just given in the previous section. In this section we omit all details.

It is easily verified that affine transformations of the form

$$(3.1) \quad \begin{aligned} x_1 &\mapsto qx_1, \\ x_2 &\mapsto r^2(x_2 - 2sx_3 + t), \\ x_3 &\mapsto r(x_3 + s), \\ x_4 &\mapsto qr^2(x_4 + s^2x_1 + tx_1), \end{aligned}$$

where  $q > 0$ ,  $r \in \mathbb{R}^*$ ,  $s, t \in \mathbb{R}$ , form a group and that this group acts transitively on

$$\{x \in \mathbb{R}^4 : x_4 > x_1x_2 + x_1x_3^2 \text{ and } x_1 > 0\}$$

and

$$\{x \in \mathbb{R}^4 : x_4 < x_1x_2 + x_1x_3^2 \text{ and } x_1 > 0\}.$$

It follows that  $C^>$  and  $C^<$  are holomorphically homogeneous. Neither has a bounded realisation since the complex lines  $\{z \in \mathbb{C}^4 : z_2 = z_3 = 0, z_4 = \pm 1\}$  lie entirely in  $C^>$  and  $C^<$ , respectively.

As in Section 2, to determine the full holomorphic automorphism groups of these domains, it suffices to know the local CR isotropy of the real Levi-indefinite hypersurface

$$\tilde{\Gamma} := \{z \in \mathbb{C}^4 : x_4 = x_1x_2 + x_1x_3^2 \text{ and } x_1 > 0\}$$

near the basepoint  $(1, 0, 0, 0)$ . Already we have an isotropy transformation coming from the affine transformations (3.1) of  $\mathbb{R}^4$ , namely

$$\begin{aligned} z_1 &\mapsto z_1, \\ z_2 &\mapsto r^2z_2, \\ z_3 &\mapsto rz_3, \\ z_4 &\mapsto r^2z_4, \end{aligned}$$

for  $r \in \mathbb{R}^*$ . We claim that the CR isotropy is generated, in addition, by

$$\begin{aligned} z_1 &\mapsto z_1, \\ z_2 &\mapsto z_2 + iu(z_1 - 1), \\ z_3 &\mapsto z_3, \\ z_4 &\mapsto z_4 + \frac{iu(z_1^2 - 1)}{2} \end{aligned}$$

and

$$(3.2) \quad \begin{aligned} z_1 &\mapsto z_1, \\ z_2 &\mapsto z_2 + \frac{(e^{2i\theta} - 1)z_3^2}{2}, \\ z_3 &\mapsto e^{i\theta}z_3, \\ z_4 &\mapsto z_4, \end{aligned}$$

where  $u, \theta \in \mathbb{R}$ . To see this, we check that the holomorphic change of variables

$$\begin{aligned} z_1 &= w_1 + 1, \\ z_2 &= w_2 - \frac{iw_1w_4}{10} - \frac{2w_3^2}{(w_1 + 2)^2}, \\ z_3 &= \frac{2w_3}{w_1 + 2}, \\ z_4 &= -iw_4 + w_2 + \frac{w_1(10w_2 - i(w_1 + 2)w_4)}{20} \end{aligned}$$

takes the equation of the surface into Chern-Moser normal form:–

$$\operatorname{Im} w_4 = 10 \operatorname{Re} \frac{4w_3\overline{w_3}(1 + w_1) + 2w_2w_1\overline{w_1} + \overline{w_2}w_1^2\overline{w_1} + 4\overline{w_2}w_1 + 2\overline{w_2}w_1^2}{(2 + w_1)(2 + \overline{w_1})(20 - w_1\overline{w_1})}$$

and that, in these coordinates, the CR isotropy is linear:–

$$\begin{aligned} w_1 &\mapsto w_1, \\ w_2 &\mapsto r^2(w_2 + iuw_1), \\ w_3 &\mapsto re^{i\theta}w_3, \\ w_4 &\mapsto r^2w_4, \end{aligned}$$

for  $r \in \mathbb{R}^*$ ,  $u, \theta \in \mathbb{R}$ . In the original coordinates we obtain precisely the stated isotropy. With the remaining affine symmetries ((3.1) with  $r = 1$ ) and imaginary translations, we have now found the full holomorphic automorphism group. As a convenient basis for the corresponding Lie algebra of holomorphic vector fields we may take

$$Z_1 := z_1 \frac{\partial}{\partial z_1} + z_4 \frac{\partial}{\partial z_4}, \quad Z_2 := -2z_3 \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3}, \quad Z_3 := \frac{\partial}{\partial z_2} + z_1 \frac{\partial}{\partial z_4},$$

$$Z_4 := i \frac{\partial}{\partial z_1}, \quad Z_5 := i \frac{\partial}{\partial z_2}, \quad Z_6 := i \frac{\partial}{\partial z_3}, \quad Z_7 := i \frac{\partial}{\partial z_4},$$

$$Z_8 := 2z_2 \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_3} + 2z_4 \frac{\partial}{\partial z_4}, \quad Z_9 := 2iz_1 \frac{\partial}{\partial z_2} + iz_1^2 \frac{\partial}{\partial z_4},$$

$$Z_{10} := -iz_3^2 \frac{\partial}{\partial z_2} + iz_3 \frac{\partial}{\partial z_3}$$

with commutation relations

$[Z_i, Z_j]$	$Z_1$	$Z_2$	$Z_3$	$Z_4$	$Z_5$	$Z_6$	$Z_7$	$Z_8$	$Z_9$	$Z_{10}$
$Z_1$		0	0	$-Z_4$	0	0	$-Z_7$	0	$Z_9$	0
$Z_2$			0	0	0	$2Z_5$	0	$Z_2$	0	$Z_6$
$Z_3$				$-Z_7$	0	0	0	$2Z_3$	0	0
$Z_4$					0	0	0	0	$-2Z_3$	0
$Z_5$						0	0	$2Z_5$	0	0
$Z_6$							0	$Z_6$	0	$-Z_2$
$Z_7$								$2Z_7$	0	0
$Z_8$									$-2Z_9$	0
$Z_9$										0
$Z_{10}$										

The rest of the discussion follows exactly the corresponding discussion in Section 2 (except that  $i$  is spanned by  $Z_8, Z_9 - 2Z_5 - Z_7, Z_{10}$ ) and we have proved:–

**Theorem 3.1.** *Each of the domains  $C^>, C^< \subset \mathbb{C}^4$  is holomorphically homogeneous, does not have a bounded realisation, and is not equivalent to a nil-ball.*

Finally notice that  $C^>$  and  $C^<$  are holomorphically distinguished from  $D^>$  and  $D^<$ , respectively, by their admitting a holomorphic circle action (3.2).

## REFERENCES

- [B] V. K. Beloshapka, *On the dimension of the group of automorphisms of an analytic hypersurface (translated from Russian)*, Math. USSR-Izv. **14** (1980) 223–245.
- [BSW] D. Burns, Jr., S. Shnider, and R. O. Wells, Jr., *Deformations of strictly pseudoconvex domains*, Invent. Math. **46** (1978) 237–253.
- [C] É. Cartan, *Sur les domaines bornés homogènes de l’espace de  $n$  variables complexes*, Abh. Math. Sem. Univ. Hamburg **11** (1935) 116–162.
- [CM] S. S. Chern and J. K. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974) 219–271.
- [DKR] B. M. Doubrov, B. P. Komrakov, and M. Rabinovich, *Homogeneous surfaces in the three-dimensional affine geometry*, Geometry and Topology of Submanifolds VIII (F. Dillen et al., eds.), World Scientific, 1996, 168–178.
- [EE1] M. G. Eastwood and V. V. Ezhov, *On affine normal forms and a classification of homogeneous surfaces in affine three-space*, Geom. Dedicata **77** (1999) 11–69.
- [EE2] M. G. Eastwood and V. V. Ezhov, *Homogeneous hypersurfaces with isotropy in affine four-space*. Tr. Mat. Inst. Steklova **235** (2001) 57–70.
- [EI] M. G. Eastwood and A. V. Isaev, *Examples of unbounded homogeneous domains in complex space*, arXiv:math.CV/0304102.
- [E1] V. V. Ezhov, *Linearization of stability group for a class of hypersurfaces (translated from Russian)*, Russian Math. Surveys **41** (1986) 203–204.

- [E2] V. V. Ezhov, *On the linearization of automorphisms of a real analytic hypersurface (translated from Russian)*, Math. USSR-Izv. **27** (1986) 53–84.
- [I] A. V. Isaev, *Global properties of spherical tube hypersurfaces*, Indiana Univ. Math. J. **42** (1993) 179–213.
- [L1] A. V. Loboda, *On local automorphisms of real-analytic hypersurfaces (translated from Russian)*, Math. USSR-Izv. **18** (1982) 537–559.
- [L2] A. V. Loboda, *Each holomorphically homogeneous tube in  $\mathbb{C}^2$  has an affine-homogeneous base (translated from Russian)*, Siberian Math. J. **42** (2001) 1111–1114.
- [N] K. Nakajima, *Homogeneous hyperbolic manifolds and homogeneous Siegel domains*, J. Math. Kyoto Univ. **25** (1985) 269–291.
- [NS1] K. Nomizu and T. Sasaki, *Affine Differential Geometry*, Cambridge University Press, 1994.
- [P1] R. Penney, *Harmonic analysis on homogeneous domains in  $\mathbb{C}^n$* , Lie Group Representations III (R. Herb et al., eds.), Lecture Notes in Math. 1077, Springer, 1984, 359–374.
- [P2] R. Penney, *The structure of rational homogeneous domains in  $\mathbb{C}^n$* , Ann. of Math. **126** (1987) 389–414.
- [P3] R. Penney, *Homogeneous Koszul manifolds in  $\mathbb{C}^n$* , J. Differential Geom. **36** (1992) 591–631.
- [P-S] I. Pyatetskii-Shapiro, *Automorphic Functions and the Geometry of Classical Domains (translated from Russian)*, Gordon and Breach, 1969.
- [VK] A.G. Vitushkin and N. G. Kruzhilin, *Description of the automorphism groups of real hypersurfaces of complex space (Russian)*, Investigations in the theory of the approximation of functions, Akad. Nauk SSSR Bashkir. Filial, Otdel Fiz. Mat., Ufa, 1987, 26–69.

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